BIRATIONAL GEOMETRY OF THE TWOFOLD SYMMETRIC PRODUCT OF A HIRZEBRUCH SURFACE VIA SECANT MAPS

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Abstract. In this paper, extending some ideas of Fano in [7] and of the first and last author in [3], we study the birational geometry of the Hilbert scheme of 0–dimensional subschemes of length 2 of a rational normal scroll $\mathbb{F}_n$. This fourfold has three elementary contractions associated to the three faces of its nef cone. We study natural projective realizations of these contractions. In particular, given a smooth rational normal scroll $S_{a,b}$ of degree r in $\mathbb{P}^{r+1}$ with $1 \leqslant a \leqslant b$ and $a+b = r$, i.e., $S_{a,b} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ embedded in $\mathbb{P}^{r+1}$ with its $\mathcal{O}(1)$ line bundle (from an abstract viewpoint $S_{a,b} \cong \mathbb{F}_{b-a}$), we consider the variety $X_{a,b} \subset \mathbb{G}(1, r+1)$ described by all lines that are secant or tangent to $S_{a,b}$. The variety $X_{a,b}$ is the image of some of the aforementioned contractions, it is smooth if $a > 1$, and it is singular at a unique point if $a = 1$. We compute the degree of $X_{a,b}$ and the local structure of the singularity of $X_{a,b}$ when $a = 1$. Finally we discuss in some detail the case $r = 4$, originally considered by Fano in [7], because the smooth hyperplane sections of $X_{2,2}$ and $X_{1,3}$ are the Fano 3–folds that appear as number 16 in the Mori–Mukai list of Fano 3–folds with Picard number 2. We prove that any smooth hyperplane section of $X_{2,2}$ is also a hyperplane section of $X_{1,3}$, and we discuss the GIT–stability of the smooth hyperplane sections of $X_{1,3}$ where G is the subgroup of the projective automorphisms of $X_{1,3}$ coming from the ones of $S_{1,3}$.

1. Introduction

In 1949 Fano published his last paper on 3-folds where he constructed a smooth 3-fold of degree 22 in a projective space of dimension 13 with canonical curve section, [7]. The first and last authors recently revised in [3] this paper with the dual purpose of providing a detailed proof of all Fano's claims and of setting up the construction in modern language. This 3-fold is in fact a Fano manifold in modern sense, that is its anticanonical divisor is very ample, and it was neglected by next mathematicians. In [3] it was denoted as *Fano's last Fano* (FlF) and it was pointed out that it is the number 16 in the Mori–Mukai list of Fano 3–folds with Picard number 2 (see [13]).

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Fano obtained his FlF has an hyperplane section of the 4-fold contained in the Grassmannian of lines in $\mathbb{P}^5$, which is the union of all secants (and tangents) lines of a general rational normal scroll of degree four in $\mathbb{P}^5$. His ingenuous geometric construction fits particularly well with the algebraic concept of Hilbert schemes of points on a surface, developed few years later, and could be subject of many generalizations. In this paper we propose some of them.

Consider the Hilbert scheme of length two 0–dimensional subschemes of the Hirzebruch surfaces $\pi : \mathbb{F}_n \longrightarrow \mathbb{P}^1$, which we denote by $\mathbb{F}_n[2]$. This abstract scheme was studied primarily by Fogarty, who proved that it is a smooth variety of dimension 4 and Picard rank 3, [9].

Subsequently it was noted, see [5], that $\mathbb{F}_n[2]$ has three elementary contractions associated to the three faces of its nef cone. One is the map $\phi_n : \mathbb{F}_n[2] \longrightarrow \mathbb{F}_n(2)$, which is the obvious map from the Hilbert scheme to the Chow scheme, or symmetric product. The other two, which we denote by $\phi_{n,i} : \mathbb{F}_n[2] \longrightarrow Z_{n,i}$ for $i = 1, 2$, are respectively described as follows. The map $\phi_{n,1}$ contracts the divisor of all pairs of points on a fiber of π to a smooth, rational curve $\Gamma \subset Z_{n,1}$; one can see, Proposition (2.2), that $Z_{n,1}$ is smooth and $\phi_{n,1}$ is the blow–up of Γ. For $n \geqslant 1$, $\phi_{n,2}$ contracts the surface $\mathcal{E}_n \cong \mathbb{P}^2$ of all pairs of points on E, the section of π with self intersection $-n$, to a point $\mathfrak{p}$. The map $\phi_{0,2}$ on the other side is as $\phi_{0,1}$, i.e., it is a smooth blow-down obtained contracting all pairs of points on a fiber of the other ruling of $\mathbb{F}_0$.

Note that the exceptional loci of $\phi_{n,1}$ and $\phi_{n,2}$ are disjoint, so the two contractions can be performed independently obtaining a morphism $\psi_n : \mathbb{F}_n[2] \longrightarrow X_n$ into a variety X_n which is smooth for $n = 0$, whereas for $n > 1$ it has a singular point in $\mathfrak{p}$.

To obtain a *projective realization* of the above varieties and morphisms we consider the smooth rational normal scrolls $S_{a,b}$ of degree r in $\mathbb{P}^{r+1}$. More precisely $S_{a,b} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$, with $1 \leqslant a \leqslant b$ and $a + b = r$, embedded in $\mathbb{P}^{r+1}$ with its $\mathcal{O}(1)$ line bundle. From an abstract viewpoint one has $S_{a,b} \cong \mathbb{F}_{b-a}$.

Take the morphism

$$\gamma_{a,b} : S_{a,b}[2] \cong \mathbb{F}_{b-a}[2] \longrightarrow \mathbb{G}(1, r+1) \subset \mathbb{P}^{\frac{r(r+3)}{2}}$$

to the Grassmannian of lines in $\mathbb{P}^{r+1}$ in its Plücker embedding, acting in the following way: each 0–dimensional length 2 scheme $\mathfrak{y} \in S_{a,b}[2]$ is mapped by $\gamma_{a,b}$ to the line $\ell_{\mathfrak{y}} := \langle \mathfrak{y} \rangle$ spanned in $\mathbb{P}^{r+1}$ by $\mathfrak{y}$. We call $\gamma_{a,b}$ the *secant map* of $S_{a,b}$ and we denote its image by $X_{a,b}$.

The case $a = b = 1$ is trivial, since $X_{1,1} = \mathbb{G}(1, 3)$. On the other hand $X_{2,2}$ is smooth and its is the 4-fold considered by Fano, whose smooth hyperplane sections are FlFs.

The secant map

$$\gamma_{a,b} : S_{a,b}[2] \longrightarrow X_{a,b}$$

is a projective realization of the map $\phi_{b-a,1} : \mathbb{F}_{b-a}[2] \longrightarrow Z_{b-a,1}$ if $a \geqslant 2$. If $a = 1$ it is projective realization of the map $\psi_{b-1} : \mathbb{F}_{b-1}[2] \longrightarrow X_{b-1}$.

In the first part of the paper we tackle two questions. First we compute the degree of $X_{a,b}$ in the Plücker embedding in $\mathbb{P}^{\frac{r(r+3)}{2}}$; namely we prove that $\deg(X_{a,b}) = 3r^2 - 8r + 6$ (see Theorem (3.6)). The formula agrees with the computation in [6] of the degree of the image of the secant map for a general surface S; however the assumption in [6] is quite stronger, namely the surface S has to be embedded by a 3-very ample line bundle.

Then we prove that $X_{a,b}$ is smooth, except for $X_{1,r-1}$, in which case it has a unique singular point at $\mathfrak{p}$, whose tangent cone is the cone over a smooth threefold linear section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{r-2}$ in $\mathbb{P}^{3r-4}$ (see Theorem (3.7)).

In Section 4 we discuss the case $r = 4$ which gives rise to the FlF. The variety $X_{2,2}$ is, as we said, smooth while $X_{1,3}$ has a singular point. They are both Fano in Fano's sense, that is their general curve section is a smooth and canonical of genus 12. Their general hyperplane section is a FlF and moreover any FLF is a hyperplane section of a $X_{1,3}$ (Remark(4.2).

The surface $S_{1,3}$ is a specialization of $S_{2,2}$ and therefore $X_{1,3}$ is a specialization of $X_{2,2}$. Most of the section is devoted to describe in different ways how this specialization takes place. In particular in Remark(4.1) we give a nice geometric description of it via a series of birational maps, starting from the specialization of the smooth quadric $S_{1,1}$ to the singular quadric $S_{0,2}$. We use the facts that $X_{2,2}$ can be obtained as the blown-up of $X_{1,1} = \mathbb{G}(1,2)$ along a conic not contained in a plane of $\mathbb{G}(1,3)$. These examples could be useful to test some conjectures on K-stability of deformations of Fano Varieties.

Finally in Section 5 we approach the question of defining an appropriate moduli space for the FlFs, studying when a hyperplane section of $X_{1,3}$ is G-stable, where G is the subgroup of the projective automorphisms of $X_{1,3}$ coming from the ones of $S_{1,3}$.

2. Hirzebruch surfaces and their twofold symmetric product

In this section we will consider Hirzebruch surfaces $\mathbb{F}_n$, with their structure morphism $\pi : \mathbb{F}_n \to \mathbb{P}^1$. We denote by F a fibre of π and by E a section of π such that $E^2 = -n$ (this section is unique if $n > 0$). We will abuse notation and denote by F and E also the line bundles on $\mathbb{F}_n$ associated to F and E. We will keep this notation throughout the paper.

We denote by $\mathbb{F}_n[2]$ the Hilbert scheme of length two 0–dimensional subschemes of $\mathbb{F}_n$, that is a smooth variety of dimension 4. We also have the symmetric product $\mathbb{F}_n(2) = \mathbb{F}_n \times \mathbb{F}_n/\mathbb{Z}_2$ and the obvious morphism $f_n : \mathbb{F}_n[2] \longrightarrow \mathbb{F}_n(2)$.

Let L be a line bundle on $\mathbb{F}_n$. Then $L^{\boxtimes 2} := L \boxtimes L$ is a line bundle on $\mathbb{F}_n \times \mathbb{F}_n$ invariant under the action of $\mathbb{Z}_2$, so $L^{\boxtimes 2}$ descends to a line bundle $L(2)$ on $\mathbb{F}_n(2)$. We denote by $L[2]$ the line bundle $f_n^*(L(2))$ on $\mathbb{F}_n[2]$. In particular we can consider the line bundles $F[2]$ and $E[2]$ on $\mathbb{F}_n[2]$. On $\mathbb{F}_n[2]$ there is a third relevant divisor (or line bundle), namely the *diagonal*, i.e., the set B of all non–reduced subschemes in $\mathbb{F}_n[2]$, that is contracted to the 2–dimensional diagonal of $\mathbb{F}_n(2)$ by f_n. It turns out that B is divisible by 2 in $\mathrm{Pic}(\mathbb{F}_n[2])$ so that we can consider the line bundle $\frac{B}{2}$. We note that $F[2]$ and $E[2]$ and $\frac{B}{2}$ are a basis of $\mathrm{Pic}(\mathbb{F}_n[2])$ (see [9]).

The following result is contained in [5, Thm. 1]:

Theorem 2.1. *The nef cone of* $\mathbb{F}_n[2]$ *is the convex hull of the rays generated by*

$$F[2], \quad E[2] + nF[2], \quad E[2] + (n+1)F[2] - \frac{B}{2}.$$

The three faces of this cone correspond to contractions of extremal rays (i.e., the relative Picard group is $\mathbb{Z}$), and precisely the contractions in question are described in the following diagram

$$\begin{array}{ccccc} Z_{n,1} & \xleftarrow{\phi_{n,1}} & \mathbb{F}_n[2] & \xrightarrow{\phi_{n,2}} & Z_{n,2} \\ & & \Big\downarrow{\scriptstyle \phi_n} & & \\ & & \mathbb{F}_n(2) & & \end{array}$$

where:
(i) $\phi_n = f_n$ is the map determined by the face bounded by $F[2]$ and $E[2] + nF[2]$;
(ii) $\phi_{n,1}$ is the map determined by the face bounded by $F[2]$ and $E[2]+(n+1)F[2]-\frac{B}{2}$;
(iii) $\phi_{n,2}$ is the map determined by the face bounded by $E[2] + nF[2]$ and $E[2] + (n+1)F[2] - \frac{B}{2}$.

Let $F(2)$ be the curve in $\mathbb{F}_n[2]$ described by the pairs of points of a g^1_2 on a curve F. One has $F(2) \cdot F[2] = F(2) \cdot (E[2] + (n+1)F[2] - \frac{B}{2}) = 0$ (see [5, p. 23]), hence $\phi_{n,1}$ contracts the divisor $\mathcal{F}_n$ on $\mathbb{F}_n[2]$ described by all pairs of points on a curve in $|F|$, to a curve $\Gamma \subset Z_{n,1}$ isomorphic to $\mathbb{P}^1$. If $n = 0$, similarly $\phi_{0,2}$ contracts the divisor $\mathcal{F}'_0$ on $\mathbb{F}_0[2]$ described by all pairs of points on a curve in $|E|$, to a curve $\Gamma' \subset Z_{0,2}$ isomorphic to $\mathbb{P}^1$. If $n > 0$, let $E(2)$ be the curve in $E[2]$ described by all pairs of points of a g^1_2 instead. Then $E(2) \cdot (E[2] + nF[2]) = E(2) \cdot \left(E[2] + (n+1)F[2] - \frac{B}{2}\right) = 0$ (see again [5, p. 23]), hence $\phi_{n,2}$ contracts to a point $\mathfrak{p}$ the surface $\mathcal{E}_n \cong \mathbb{P}^2$ of all pairs of points on E.

Proposition 2.2. *The 4–dimensional variety* $Z_{n,1}$ *is smooth and* $\mathbb{F}_n[2]$ *is the blow–up of* $Z_{n,1}$ *along the smooth rational curve* Γ. *If* $n = 0$*, the same happens for* $Z_{0,2}$.

Proof. One has $E[2] \cdot F(2) = 1$ (see [5, p. 23]), hence $\phi_{n,1}$ is the contraction of the ray R generated by $F(2)$. Now notice that

$$K_{\mathbb{F}_n[2]} \equiv K_{\mathbb{F}_n}[2] = -2E[2] - (n+2)F[2]$$

(see [5, Proof of Cor. 3]). Hence

$$\ell(R) := \inf\{-K_{\mathbb{F}_n[2]} \cdot C | C \in R\} = 2.$$

Since $\ell(R)$ equals the dimension of the fibres of $\phi_{n,1}$, we can apply [2, Thm. 5.1], which implies the first assertion. The final assertion is clear. □

If $n > 1$, since the morphism $\phi_{n,2}$ is a small (hence crepant) contraction, the variety $Z_{n,2}$ cannot be smooth at $\mathfrak{p}$, whereas it is smooth off $\mathfrak{p}$. We do not dwell now on the nature of the singularity of $Z_{n,2}$ at $\mathfrak{p}$, but we will return on this later. For the time being we notice that the loci $\mathcal{F}_0$ and $\mathcal{F}'_0$, and the loci $\mathcal{F}_n$ and $\mathcal{E}_n$ for

$n > 1$, contracted by $\phi_{n,1}$ and $\phi_{n,2}$ are disjoint. So the contractions of $\mathcal{F}_n$ and $\mathcal{E}_n$ can be performed independently obtaining a morphism ψ_n appearing in the following commutative diagram

$$\begin{array}{ccc} \mathbb{F}_n[2] & \xrightarrow{\phi_{n,1}} & Z_{n,1} \\ {\scriptstyle \phi_{n,2}}\downarrow & \searrow^{\psi_n} & \downarrow{\scriptstyle \psi_{n,1}} \\ Z_{n,2} & \xrightarrow[\psi_{n,2}]{} & X_n \end{array} \tag{2.1}$$

The variety X_n is smooth for $n = 0$, whereas for $n > 1$ it is singular at the point $\mathfrak{q} = \psi_{n,2}(\mathfrak{p})$, where X_n has the same singularity as $Z_{n,2}$.

3. The secant map

Let us consider now a linearly normal smooth rational normal scroll $S_{a,b}$ of degree r in $\mathbb{P}^{r+1}$, where

$$S_{a,b} = \mathbb{P}(O_{\mathbb{P}^1}(a) \oplus O_{\mathbb{P}^1}(b))$$

with $1 \leqslant a \leqslant b$ and $a + b = r$, embedded in $\mathbb{P}^{r+1}$ with its $O(1)$ line bundle. From an abstract viewpoint one has $S_{a,b} \cong \mathbb{F}_{b-a}$. We will consider the morphism

$$\gamma_{a,b} : S_{a,b}[2] \cong \mathbb{F}_{b-a}[2] \longrightarrow \mathbb{G}(1, r+1) \subset \mathbb{P}^{\frac{r(r+3)}{2}}$$

to the Grassmannian of lines in $\mathbb{P}^{r+1}$ in its Plücker embedding, acting in the following way: each 0–dimensional lenght 2 scheme $\mathfrak{y} \in S_{a,b}[2]$ is mapped by $\gamma_{a,b}$ to the line $\ell_{\mathfrak{y}} := \langle \mathfrak{y} \rangle$ spanned in $\mathbb{P}^{r+1}$ by $\mathfrak{y}$. We call $\gamma_{a,b}$ the *secant map* of $S_{a,b}$. Its image is denoted by $X_{a,b}$.

Lemma 3.1. *Let $Z \subset \mathbb{P}^r$ be a closed subscheme whose ideal is generated by quadrics.*

Ler $\mathfrak{y}$ be a length 2 subscheme of Z and let $l_{\mathfrak{y}}$ the line $\langle \mathfrak{y} \rangle$ in $\mathbb{P}^r$. Then the intersection of $l_{\mathfrak{y}}$ with Z consists of $\mathfrak{y}$ unless $l_{\mathfrak{y}}$ is contained in Z. In particular given two distinct subschemes $\mathfrak{y}, \mathfrak{z}$ of length 2 in Z, $l_{\mathfrak{y}} = l_{\mathfrak{z}}$ if and only if there is a line l contained in Z such that both $\mathfrak{y}$ and $\mathfrak{z}$ are subschemes of l.

Proof. This is an immediate consequence of the fact that the ideal of Z is generated by quadrics. □

Note that lemma 3.1 applies to $S_{a,b}$ whose ideal is generated by quadrics.

Now we have two main questions to deal with:

(a) what is the degree of $X_{a,b}$ in the Plücker embedding in $\mathbb{P}^{\frac{r(r+3)}{2}}$?
(b) what are the smooth points and the singular points of $X_{a,b}$, if any?

3.1. **The degree of $X_{a,b}$.** To compute the degree of $X_{a,b}$ we extend a beautiful argument by Fano (see [7]), revisited in [3].

We compute the degree of the 4–dimensional variety $X_{a,b}$ as the degree of a surface obtained cutting $X_{a,b}$ with two independent hyperplanes in $\mathbb{P}^{\frac{r(r+3)}{2}}$. We will consider two special hyperplane sections of $\mathbb{G}(1, r+1)$ given by the lines that intersect two linear subspaces σ_1, σ_2 of codimension 2 in $\mathbb{P}^{r+1}$ that are in special position, i.e., they span a general hyperplane σ of $\mathbb{P}^{r+1}$ and are general there, and therefore intersect in a general subspace π of codimension 3 in $\mathbb{P}^{r+1}$.

Then the set of lines in the Grassmanian that intersect both σ_1, σ_2 breaks up in two codimension 2 Schubert cycles: the lines contained in the hyperplane σ and the lines intersecting the subspace π. We denote by $X^{\sigma}_{a,b}$ the subvariety of $X_{a,b}$ consisting of the lines contained in σ and by $X^{\pi}_{a,b}$ the subvariety of $X_{a,b}$ consisting of the lines intersecting π.

Lemma 3.2. *One has*

$$\deg(X_{a,b}) = \deg(X^{\sigma}_{a,b}) + \deg(X^{\pi}_{a,b}).$$

Proof. The assertion will follow once we prove that $X^{\sigma}_{a,b} \cup X^{\pi}_{a,b}$, with its reduced structure, is the scheme theoretical intersection of $X_{a,b}$ with the aforementioned two hyperplanes. To prove this, we will make a computation in coordinates and we will imitate a similar computation made in [3].

First of all we prove that for a general choice of σ and π, $X^{\sigma}_{a,b}$ and $X^{\pi}_{a,b}$ are both irreducible.

The assertion is trivial for $X^{\sigma}_{a,b}$ that is the secant variety of the rational curve intersection of $X_{a,b}$ with σ.

As for $X^{\pi}_{a,b}$, we consider the linear projection $f\colon S_{a,b} \to \mathbb{P}^2$ with center π. This is a degree r finite cover whose monodromy is the full symmetric group ([15, 17]). Then a dense open subset of $X^{\pi}_{a,b}$ can be identified with the pairs of points contained in any fibre of f consisting of r distinct points. Since the symmetric group is $2-$transitive this open subset is irreducible.

Next we introduce homogeneous coordinates $[x_0, \dots, x_{r+1}]$ in $\mathbb{P}^{r+1}$ so that $S_{a,b}$ has equations

$$\operatorname{rk}\begin{pmatrix} x_0 & \dots & x_{b-1} & x_{b+1} & \dots & x_r \\ x_1 & \dots & x_b & x_{b+2} & \dots & x_{r+1} \end{pmatrix} < 2.$$

Consider the line ℓ with equations $x_1 = \dots = x_b = x_{b+2} = \dots = x_{r+1} = 0$, that is a line of the ruling of $S_{a,b}$. An open neighborhood of ℓ in $\mathbb{G}(1, r+1)$ consists of all lines joining the points whose homogeneous coordinates are given by the rows of the following matrix

$$\begin{pmatrix} 1 & \xi_1 & \dots & \xi_b & 0 & \xi_{b+2} & \dots & \xi_{r+1} \\ 0 & \eta_1 & \dots & \eta_b & 1 & \eta_{b+2} & \dots & \eta_{r+1} \end{pmatrix}.$$

This tells us that

$$\xi_1, \dots, \xi_b, \xi_{b+2}, \dots, \xi_{r+1}, \eta_1, \dots, \eta_b, \eta_{b+2}, \dots, \eta_{r+1}$$

are coordinates of a chart U of $\mathbb{G}(1, r+1)$ centered at ℓ. A line t parametrized by a point of U has parametric equations of the form

$$x_0 = \lambda, x_{b+1} = \mu, x_i = \lambda\xi_i + \mu\eta_i, \quad i \in \{1, \dots, r+1\} \setminus \{b+1\}$$

with $[\lambda, \mu] \in \mathbb{P}^1$.

The intersection of t with $S_{a,b}$ is obtained by solving in λ, μ the system of equations

$$\operatorname{rk}\begin{pmatrix} \lambda & \lambda\xi_1 + \mu\eta_1 & \dots & \lambda\xi_{b-1} + \mu\eta_{b-1} & \mu & \lambda\xi_{b+2} + \mu\eta_{b+2} & \dots & \lambda\xi_r + \mu\eta_r \\ \lambda\xi_1 + \mu\eta_1 & \dots & \dots & \lambda\xi_b + \mu\eta_b & \lambda\xi_{b+2} + \mu\eta_{b+2} & \dots & \dots & \lambda\xi_{r+1} + \mu\eta_{r+1} \end{pmatrix} < 2$$

that are quadratic in λ, μ. The line t is secant (or tangent) to $S_{a,b}$ if and only if all the equations in λ, μ that we obtain in this way are proportional. One of these equations, obtained by considering the minor determined by the first and $(b+1)$–th columns is the following

$$\lambda^2 \xi_{b+2} + \lambda\mu(\eta_{b+2} - \xi_1) - \mu^2 \eta_1 = 0 \tag{3.1}$$

Suppose that for the line t one has $\eta_1 \neq 0$.

Another equation, obtained by considering the minor determined by the first and second columns, is the following

$$\lambda^2(\xi_2 - \xi_1^2) + \lambda\mu(\eta_2 - 2\xi_1\eta_1) - \mu^2\eta_1^2 = 0$$

Since this equation has to be proportional to the one in (3.1), we get

$$\xi_2 = \xi_1^2 + \eta_1\xi_{b+2}, \quad \eta_2 = \xi_1\eta_1 + \eta_1\eta_{b+2}$$

so that ξ_2, η_2 can be expressed as polynomials in $\xi_1, \eta_1, \xi_{b+2}, \eta_{b+2}$. Now we claim that also ξ_i, η_i, with $i \in \{3, \ldots, r+1\} \setminus \{b+2\}$ can be expressed as polynomials in $\xi_1, \eta_1, \xi_{b+2}, \eta_{b+2}$. This can be proved by induction on i. Indeed, assume we have proved the assertion for i. Then consider the equation

$$\lambda(\lambda\xi_{i+1} + \mu\eta_{i+1}) = (\lambda\xi_1 + \mu\eta_1)(\lambda\xi_i + \mu\eta_i).$$

Since this has to be proportional to the one in (3.1), we get

$$\xi_{i+1} = \xi_1\xi_i + \eta_i\xi_{b+2}, \quad \eta_{i+1} = \xi_i\eta_1 + \eta_i\eta_{b+2} \tag{3.2}$$

and, applying induction, from this we see that also ξ_{i+1}, η_{i+1} can be expressed as polynomials in $\xi_1, \eta_1, \xi_{b+2}, \eta_{b+2}$.

Now let $Z \subset X_{a,b} \cap U$ be the set of points with $\eta_1 = 0$. This is a proper closed subset of $X_{a,b} \cap U$. We notice that $X_{a,b} \cap U \setminus Z$ is contained in U' where U' is the closed embedding of $\mathbb{C}^4$ in $U = \mathbb{C}^{2r}$ defined by the equations (3.2) (for $i \in \{2, \ldots, r+1\} \setminus \{b+2\}$), with variables in $\mathbb{C}^4$ given by $\xi_1, \eta_1, \xi_{b+2}, \eta_{b+2}$.

Hence $X_{a,b} \cap U = \overline{(X_{a,b} \cap U) \setminus Z} \subseteq U'$. Then, since $X_{a,b}$ is irreducible of dimension 4, and $X_{a,b} \cap U$ is closed in U' which is also irreducible of dimension 4, we deduce that $X_{a,b} \cap U = U' \cong \mathbb{C}^4$ with coordinates $\xi_1, \eta_1, \xi_{b+2}, \eta_{b+2}$.

Now let us consider the two codimension 2 subspaces σ_1, σ_2 with equations

$$\sigma_1) \quad x_0 - x_1 = x_{b+1} = 0, \qquad \sigma_2) \quad x_0 - x_1 = x_{b+2} = 0$$

that intersect along the codimension 3 subspace π with equations

$$\pi) \quad x_0 - x_1 = x_{b+1} = x_{b+2} = 0 \tag{3.3}$$

and span the hyperplane σ with equation $x_0 - x_1 = 0$. The set of points in the chart U' corresponding to lines intersecting σ_1 has equation $\xi_1 = 1$, whereas the set of points in U' corresponding to lines intersecting σ_2 has equation $(\xi_1 - 1)\eta_{b+2} = \eta_1\xi_{b+2}$, so the intersection of the two sets has equations $\xi_1 = 1, \eta_1\xi_{b+2} = 0$, and this splits into two irreducible and reduced components with equations $\xi_1 = 1, \eta_1 = 0$ and $\xi_1 = 1, \xi_{b+2} = 0$. The former equations define the set of points in U' belonging to $X^{\sigma}_{a,b}$, the latter equations define the set of points in U' belonging to $X^{\pi}_{a,b}$.

In conclusion, we proved that for a specific choice of σ_1 and σ_2 in a hyperplane σ and intersecting in a codimension 3 subspace π we have that $X^{\sigma}_{a,b} \cup X^{\pi}_{a,b}$, with its reduced structure, is the scheme theoretical intersection of $X_{a,b}$ with the two hyperplane sections of lines intersecting σ_1 and σ_2. Then this is true for a general choice of σ, σ_1 and σ_2, and the assertion follows. □

To compute the degree of $X_{a,b}$ we have to compute the degrees of $X^{\sigma}_{a,b}$ and $X^{\pi}_{a,b}$. Before doing that, we need some preliminary results.

Let $C_d \subset \mathbb{P}^d$ be a rational normal curve of degree $d \geqslant 2$. Let $C_d(2) \cong \mathbb{P}^2$ be the symmetric product of C_d. We have the *secant morphism*

$$\varphi_d : C_d(2) \cong \mathbb{P}^2 \longrightarrow \mathbb{G}(1, d) \subset \mathbb{P}^{\frac{d(d+1)}{2}-1}$$

that maps a $\mathfrak{y} \in C_d(2)$ to the line $\ell_{\mathfrak{y}} = \langle \mathfrak{y} \rangle$. We note that φ_d is injective by Lemma 3.1. Let us denote by V_d the image of φ_d.

Proposition 3.3. *V_d is the $(d-1)$–*Veronese surface*, i.e., the image of the plane via the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(d-1)|$, and φ_d is an isomorphism onto its image V_d.*

Proof. In [3] the result is proved in the case $d = 4$, and the proof of the general case is similar. However we give it here for completeness.

First of all we prove that $\deg(V_d) = (d-1)^2$. To see this, consider two general linear subspaces σ_1, σ_2 of codimension 2 in $\mathbb{P}^d$, and intersect V_d with the two hyperplane sections H_1, H_2 of $\mathbb{G}(1, d)$ of points corresponding to lines intersecting both σ_1 and σ_2. The hyperplanes through σ_i, for $i = 1, 2$, cut out on C_d a linear series g_i, of dimension 1 and degree d. The number of intersection points of V_d with $H_1 \cap H_2$ equals the number of divisors in $C_d(2)$ that are contained at the same time in divisors of g_1 and g_2. This number is well known to be $(d-1)^2$ (see [1, p. 344]), as wanted.

To finish the proof, we have to show that V_d spans $\mathbb{P}^{\frac{d(d+1)}{2}-1}$. We prove this by induction of d. The assertion is trivial for $d = 2$. So, suppose we have proved it for d and let us prove it for $d + 1$. Let us consider the rational normal curve $C_{d+1} \subset \mathbb{P}^{d+1}$ of degree $d + 1$ and let $x \in C_{d+1}$ be a point. Let us project down C_{d+1} from x, obtaining a rational normal normal curve $C_d \subset \mathbb{P}^d$. The projection from x determines also an obvious rational map

$$p : V_{d+1} \dashrightarrow V_d$$

whose indeterminacy locus is a priori contained in the set of secant lines to C_{d+1} that contain x, that is the set of lines of the ruling of the cone over C_d with vertex x. This is a rational normal curve D of degree d on V_{d+1}. Then p is the projection of V_{d+1} to V_d from the subspace spanned by D, that has dimension d. By induction we know that V_d spans a $\mathbb{P}^{\frac{d(d+1)}{2}-1}$. Then V_{d+1} spans a linear space of dimension

$$\frac{d(d+1)}{2} - 1 + d + 1 = \frac{d(d+3)}{2}$$

as required. □

Corollary 3.4. *$X_{a,b}$ is linearly normal in $\mathbb{P}^{\frac{r(r+3)}{2}}$.*

Proof. On $S_{a,b}$ there are rational normal curves C_{r+1} of degree $r+1$, and therefore V_{r+1} is contained in $X_{a,b}$. Since, by Proposition 3.3, V_{r+1} spans a linear space of dimension $\frac{r(r+3)}{2}$, we are done. □

Next we want to figure out how to compute $\deg(X^{\pi}_{a,b})$, where π is a general linear subspace of codimension 3. To do this we choose a codimension 2 linear subspace τ that intersects π along a linear subspace α of codimension 4, so that $\sigma = \langle \tau, \pi \rangle$ is a hyperplane. Then we intersect $X^{\pi}_{a,b}$ with the hyperplane section of all lines intersecting τ, thus obtaining a curve $X^{\pi,\tau}_{a,b}$. Then $X^{\pi,\tau}_{a,b}$ splits in two parts:
• the curve $X^{\alpha}_{a,b}$ of lines in $X_{a,b}$ intersecting α;
• the curve $X^{\sigma,\pi}_{a,b}$ of lines in $X_{a,b}$ contained in σ and intersecting π.

Lemma 3.5. *One has*

$$\deg(X^{\pi}_{a,b}) = \deg(X^{\alpha}_{a,b}) + \deg(X^{\sigma,\pi}_{a,b}).$$

Proof. As in Lemma 3.2, the assertion will follow once we prove that $X^{\alpha}_{a,b} \cup X^{\sigma,\pi}_{a,b}$, with its reduced structure, is the scheme theoretical intersection of $X^{\pi}_{a,b}$ with the hyperplane section of the lines intersecting τ.

We first show that $X^{\alpha}_{a,b}$ and $X^{\sigma,\pi}_{a,b}$ are both irreducible for general choices of α, π, σ.

For $X^{\sigma,\pi}_{a,b}$, we notice that it consists of the secants to the rational normal curve C_r intersection of $S_{a,b}$ with σ, that intersect π, that has codimension 2 in σ. The hyperplanes in σ containing π cut out a general g^1_r on C_r. By generality the monodromy of this g^1_r is the full symmetric group. $X^{\sigma,\pi}_{a,b}$ can be identified with the effective divisors of degree 2 on C_r that are contained in divisors of the g^1_r. Since the full symmetric group is 2−transitive, the required irreducibility follows.

As for $X^{\alpha}_{a,b}$ let us consider the projection $f\colon S_{a,b} \to \mathbb{P}^3$ with center α. The image Σ of this projection is a surface with ordinary singularities with an irreducible double curve ([10, 11, 12]). $X^{\alpha}_{a,b}$ can be identified with this double curve, so we have irreducibility.

Next we go back to the local computation we made in the proof of Lemma 3.2, from which we keep the notation.

We fix π and τ to have equations

$$\pi) \qquad x_1 - x_{b+1} = x_0 = x_{b+2} = 0, \qquad \tau) \qquad x_1 - x_{b+1} = x_{b+3} = 0$$

so that

$$\sigma) \qquad x_1 - x_{b+1} = 0, \qquad \alpha) \qquad x_1 - x_{b+1} = x_0 = x_{b+2} = x_{b+3} = 0.$$

Then $X^{\pi}_{a,b}$ has equations $\eta_1 = 1, \eta_{b+2} = 0$ and the hyperplane section given by the lines intersecting τ has equation $\xi_1 \eta_{b+3} = \xi_{b+3}(\eta_1 - 1)$. So the intersection of this hyperplane with $X^{\pi}_{a,b}$ has equations

$$\eta_1 = 1, \quad \eta_{b+2} = 0, \quad \xi_1 \eta_{b+3} = 0.$$

This intersection splits in two parts, one with equations

$$\eta_1 = 1, \quad \eta_{b+2} = 0, \quad \xi_1 = 0,$$

that is irreducible and reduced and coincides with $X^{\sigma,\pi}_{a,b}$, and the other with equations

$$\eta_1 = 1, \quad \eta_{b+2} = 0, \quad \eta_{b+3} = 0. \tag{3.4}$$

Now, by (3.2) we see that $\eta_{b+3} = \eta_1\xi_{b+2} + \eta^2_{b+2} = \xi_{b+2}$, so that (3.4) is equivalent to

$$\eta_1 = 1, \quad \eta_{b+2} = 0, \quad \xi_{b+2} = 0$$

that defines an irreducible and reduced curve, that coincides with $X^{\alpha}_{a,b}$. □

We can now prove that:

Theorem 3.6. *One has* $\deg(X_{a,b}) = 3r^2 - 8r + 6$.

Proof. We assume $r \geqslant 4$, the case $r = 3$ is similar, and in fact easier, and can be left to the reader. The case $r = 2$ is trivial.

We apply Lemmas 3.2 and 3.5. First of all we compute $\deg(X^{\sigma}_{a,b})$. We note that $X^{\sigma}_{a,b}$ is nothing but the surface described by the secant lines to a general hyperplane section of $S_{a,b}$ that is a rational normal curve of degree r. Hence, by Proposition 3.3, we have $\deg(X^{\sigma}_{a,b}) = (r-1)^2$.

Next we compute $\deg(X^{\sigma,\pi}_{a,b})$. This coincides again with the degree of the surface described by the secant lines to a general hyperplane section of $S_{a,b}$. So $\deg(X^{\sigma,\pi}_{a,b}) = (r-1)^2$.

Finally, we compute $\deg(X^{\alpha}_{a,b})$. We consider the surface scroll $\Phi \subset \mathbb{P}^{r+1}$ described by the lines in $X^{\alpha}_{a,b}$. We claim that

$$\deg(X^{\alpha}_{a,b}) = \deg(\Phi).$$

Indeed, $\deg(X^{\alpha}_{a,b})$ equals the number of points in $X^{\alpha}_{a,b}$ corresponding to lines intersecting a general linear subspace τ of codimension 2, and this is exactly the number of points that τ has in common with Φ. So we need to compute $\deg(\Phi)$.

The linear subspace α of dimension $r-3$ intersects Φ along a curve C. The curve C can also be interpreted as the scheme theoretical intersection of the secant variety $\mathrm{Sec}(S_{a,b})$ of $S_{a,b}$ with α. By the generality of α, C is irreducible and reduced, so $\deg(C) = \deg(\mathrm{Sec}(S_{a,b}))$. One has $\deg(\mathrm{Sec}(S_{a,b})) = \binom{r-2}{2}$. Indeed, if β is a general codimension 5 linear space, the number of its intersection points with $\mathrm{Sec}(S_{a,b})$ is $\deg(\mathrm{Sec}(S_{a,b}))$. On the other hand this number is also the number of double points of the projection of $S_{a,b}$ from β to $\mathbb{P}^4$, that is a general proiection of $S_{a,b}$ to $\mathbb{P}^4$, and this number is $\binom{r-2}{2}$ by the double point formula. Thus we have

$$\deg(C) = \binom{r-2}{2}.$$

Now take a general hyperplane containing α. This intersects Φ along C and along a certain number δ of lines of the ruling and

$$\deg(\Phi) = \delta + \deg(C) = \delta + \binom{r-2}{2}.$$

To compute δ, consider the projection from α to $\mathbb{P}^3$. Then $S_{a,b}$ is projected to a rational scroll Σ of degree r in $\mathbb{P}^3$ with ordinary singularities and δ coincides with the degree of the double curve of Σ, so that $\delta = \binom{r-1}{2}$. Summing up

$$\deg(X^{\alpha}_{a,b}) = \deg(\Phi) = \binom{r-1}{2} + \binom{r-2}{2} = (r-2)^2.$$

In conclusion, putting all the above information together, we get

$$\deg(X_{a,b}) = 2(r-1)^2 + (r-2)^2$$

and the assertion follows. □

We note that the formula of the Theorem 3.6 agrees with the computation in [6] of the degree of the image of the secant map for a general surface. However the hypotheses in [6] do not apply to our situation.

3.2. **Local structure of $X_{a,b}$.** Now we want to study smoothness or singularity of the points of $X_{a,b}$.

The case $a = b = 1$ is trivial since, in this case $X_{1,1} = \mathbb{G}(1,3)$. So in the rest of this section we will assume $(a,b) \neq (1,1)$ so that $r = a + b > 2$.

A preliminary remark is in order. The only lines in $S_{a,b}$ are the ones of the ruling, except for $S_{1,r-1} \cong \mathbb{F}_{r-2}$, in which there is a further line s image of the negative section E of $\mathbb{F}_{r-2}$. This line, in turn, is mapped by the secant map to a point p_s of $X_{1,r-1}$.

Theorem 3.7. *If $(a,b) \neq (1,1)$, then $X_{a,b}$ is smooth, except for $X_{1,r-1}$, in which case it has the unique singular point at p_s, whose tangent cone is the cone over a smooth threefold linear section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{r-2}$ in $\mathbb{P}^{3r-4}$, hence it has multiplicity $\binom{r}{2}$.*

Proof. In the proof of Lemma 3.2 we have proved that given a line of the ruling of $S_{a,b}$, then $X_{a,b}$ is smooth at the point corresponding to that line.

Let now ℓ be a line, not contained in $S_{a,b}$, that is secant (or tangent) to $S_{a,b}$. We want to prove that $X_{a,b}$ is smooth at the point p_ℓ corresponding to ℓ. To see this, we take the intersection of $X_{a,b}$ with two suitable hyperplanes containing p_ℓ obtaining a surface and we will show that this surface is smooth at p_ℓ.

We argue as in the proof of Lemma 3.2. Consider a general codimension 3 linear subspace π and two distinct codimension 2 linear subspaces σ_1 and σ_2 containing π and intersecting in one point each, and in different points, ℓ, and therefore spanning a hyperplane σ containing ℓ. As we saw in Lemma 3.2 (from which we keep the notation), the intersection of $X_{a,b}$ with the two hyperplanes of lines intersecting σ_1 and σ_2 is the reduced union $X^{\sigma}_{a,b} \cup X^{\pi}_{a,b}$. The point p_ℓ does not sit in $X^{\pi}_{a,b}$ but sits in $X^{\sigma}_{a,b}$, which is the surface described by the secant lines to a general hyperplane section of $S_{a,b}$ that is a rational normal curve of degree r. Hence, by Proposition 3.3, $X^{\sigma}_{a,b}$ is smooth, thus $X^{\sigma}_{a,b} \cup X^{\pi}_{a,b}$ is smooth at p_ℓ as wanted.

Let us now turn to the case of $X_{1,r-1}$ and to the point p_s. We may assume that $S_{1,r-1}$ is defined by the equations

$$\operatorname{rk}\begin{pmatrix} x_0 & y_0 & y_1 & \dots & y_{r-2} \\ x_1 & y_1 & y_2 & \dots & y_{r-1} \end{pmatrix} < 2,$$

so that the line s is defined by the equations $\{y_i = 0\}_{0\leqslant i\leqslant r-1}$. An open neighborhood of s in $\mathbb{G}(1, r+1)$ consists of all lines joining the points whose homogeneous coordinates are given by the rows of the following matrix

$$\begin{pmatrix} 1 & 0 & \xi_0 & \xi_1 & \dots & \xi_{r-1} \\ 0 & 1 & \eta_0 & \eta_1 & \dots & \eta_{r-1} \end{pmatrix}$$

so that $\xi_0, \dots, \xi_{r-1}, \eta_0, \dots, \eta_{r-1}$ are coordinates of a chart U of $\mathbb{G}(1, r+1)$ centered at s. A line t parametrized by a point of U has parametric equations of the form

$$x_0 = \lambda, x_1 = \mu, y_i = \lambda\xi_i + \mu\eta_i, \quad i = 0, \dots, r-1.$$

with $[\lambda, \mu] \in \mathbb{P}^1$.

The intersection of t with $S_{1,r-1}$ is obtained by solving in λ, μ the system of equations

$$\operatorname{rk}\begin{pmatrix} \lambda & \lambda\xi_0 + \mu\eta_0 & \lambda\xi_1 + \mu\eta_1 & \dots & \lambda\xi_{r-2} + \mu\eta_{r-2} \\ \mu & \lambda\xi_1 + \mu\eta_1 & \lambda\xi_2 + \mu\eta_2 & \dots & \lambda\xi_{r-1} + \mu\eta_{r-1} \end{pmatrix} < 2.$$

The line t is secant (or tangent) to $S_{a,b}$ if and only if all the degree 2 equations in λ, μ that we obtain in this way are proportional. Let us consider only the equations coming from the minors including the first column. They have the form

$$\lambda^2\xi_{i+1} + \lambda\mu(\eta_{i+1} - \xi_i) - \mu^2\eta_i = 0, \quad i = 0, \dots, r-2.$$

hence the proportionality is given by the equations

$$\operatorname{rk}\begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_{r-1} \\ \eta_1 - \xi_0 & \eta_2 - \xi_1 & \dots & \eta_{r-1} - \xi_{r-2} \\ \eta_0 & \eta_1 & \dots & \eta_{r-2} \end{pmatrix} < 2 \tag{3.5}$$

that define the cone over a linear three dimensional section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{r-2}$ in $\mathbb{P}^{3r-4}$. □

Remark 3.8. Suppose $r \geqslant 3$. Recalling what has been said in §2, we have that the secant map

$$\gamma_{a,b} : S_{a,b}[2] \longrightarrow X_{a,b}$$

is:

- a projective realization of the map $\phi_{b-a,1} : \mathbb{F}_{b-a}[2] \longrightarrow Z_{b-a,1}$, if $a \geqslant 2$;
- a projective realization of the map $\psi_{b-a} : \mathbb{F}_{b-a}[2] \longrightarrow X_{b-a}$, if $a = 1$.

In particular we have that if $2 \leqslant a \leqslant b$ and $2 \leqslant a' \leqslant b'$ and $n := b - a = b' - a'$, then $X_{a,b}$ is isomorphic to $X_{a',b'}$ and both are isomorphic to $Z_{n,1}$.

This tells us what is the local nature of the singularity of the variety $Z_{n,2}$, image of the morphism $\phi_{n,2} : \mathbb{F}_n[2] \longrightarrow Z_{n,2}$, at the point $\mathfrak{p}$ image of the surface $\mathcal{E}_n \cong \mathbb{P}^2$. It suffices to fix $a = 1, b = n+1$, so that $r = a + b = n + 2$ and we have that $Z_{n,2}$ has at $\mathfrak{p}$ the same singularity as $X_{1,r-1}$ at the point $\mathfrak{p}_s$ corresponding to the line s on $S_{1,r-1}$ not belonging to the ruling. So the tangent cone there is the cone over a threefold linear section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^n$ in $\mathbb{P}^{3n+2}$, hence it has multiplicity $\binom{n+2}{2}$.

If $r = 2$, then $a = b = 1$, $S_{1,1} \cong \mathbb{F}_0$ is a smooth quadric in $\mathbb{P}^3$, and the secant map is a projective realization of the map ψ_0 appearing in diagram (2.1), so that $X_0 = X_{1,1}$ coincides with $\mathbb{G}(1,3)$, that is a smooth quadric in $\mathbb{P}^5$. The varieties $Z_{0,1}$ and $Z_{0,2}$ are the blow–ups of $\mathbb{G}(1,3)$ along the two conics Γ and Γ' that correspond to the two rulings of $S_{1,1}$, and $\mathbb{F}_0[2]$ is the simoultaneous blow–up of these two conics with exceptional divisors $\mathcal{F}_0$ and $\mathcal{F}'_0$.

4. The case $r = 4$

The case of $S_{2,2}$ has been studied by Fano in [7] and, in recent times, in [3]. It turns out that $X_{2,2} \subset \mathbb{P}^{14}$ is a smooth Fano 4–fold of degree 22, of index 2 with canonical curve sections of genus 12. Any smooth hyperplane section of $X_{2,2}$ is a Fano 3–fold of index 1, that has been called in [3] a *Fano's last Fano* (FlF).

The surface $S_{1,3}$ is a specialization of $S_{2,2}$ and therefore $X_{1,3}$ is a specialization of $X_{2,2}$. The variety $X_{1,3} \subset \mathbb{P}^{14}$ has still degree 22 but is no longer smooth, since it has an isolated singular point of multiplicity 6, with tangent cone the cone over a 3–fold linear section of $\mathbb{P}^2 \times \mathbb{P}^2$. However the general curve section of $X_{1,3}$ is still smooth and canonical of genus 12. So $X_{1,3}$ is a weak Fano variety, the general threefold section of $X_{1,3}$ is a smooth Fano threefold of index 1, and it is still a FlF.

To better understand in which way the specialization of $X_{2,2}$ to $X_{1,3}$ takes place, it is useful to briefly recall some of the results in [3].

We have $S_{2,2} \cong \mathbb{F}_0$ and, as we saw in Remark 3.8, $X_{2,2}$ coincides with the variety $Z_{0,1}$, that is the blow–up of the quadric 4–fold $\mathbb{G}(1,3)$ along the conic Γ' corresponding to the ruling $|E|$ of $S_{1,1} \cong \mathbb{F}_0$. Note that Γ' does not belong to a plane contained in $\mathbb{G}(1,3)$.

It has been proved in [3] that the projective realization of this is the fact that $X_{2,2} \subset \mathbb{P}^{14}$ is the image of the quadric $\mathbb{G}(1,3)$ via the rational map determined by the linear system $|\mathcal{I}_{\Gamma',\mathbb{G}(1,3)}(2)|$ of quadric sections of $\mathbb{G}(1,3)$ passing through the conic Γ' that does not belong to a plane contained in $\mathbb{G}(1,3)$. As a consequence we have that the general FlF is the blow–up along a conic of a smooth complete intersection of type $(2,2)$ in $\mathbb{P}^5$. This variety appears as the number 16 in the Mori–Mukai list of Fano 3–folds with Picard number 2 (see [13, Table 2]).

Remark 4.1. If $r = 2a \geqslant 4$, then $X_{a,a}$ is isomorphic to $X_{2,2}$ hence it is isomorphic to the blow–up $\tilde{\mathbb{G}}$ of $\mathbb{G}(1,3)$ along the conic Γ' as above. If H denotes the strict transform on $\tilde{\mathbb{G}}$ of a general hyperplane section of $\mathbb{G}(1,3)$ and $\mathcal{E}$ is the exceptional divisor in $\tilde{\mathbb{G}}$ over the blown–up conic, one sees that the isomorphism of $\tilde{\mathbb{G}}$ to $X_{a,a}$ is given by the map determined by the linear system $|aH - (a-1)\mathcal{E}|$. Indeed, one computes

$$H^4 = 2, \quad H^3 \cdot \mathcal{E} = H^2 \cdot \mathcal{E}^2 = 0, \quad H \cdot \mathcal{E}^3 = 2, \quad \mathcal{E}^4 = 6$$

hence

$$(aH - (a-1)\mathcal{E})^4 = 12a^2 - 16a + 6 = 3r^2 - 8r + 6 = \deg(X_{a,a}).$$

To compute $h = \dim(|aH - (a-1)\mathcal{E}|)$, we first notice that the linear system of hypersurfaces of $\mathbb{P}^5$ of degree a having multiplicity $a-1$ along a smooth conic, is computed to be $5\frac{(a+1)a}{2}$ (the computation can be left to the reader). So $5\frac{(a+1)a}{2} - (h+1)$ is the dimension of the linear system of hypersurfaces of degree $a-2$ in $\mathbb{P}^5$ that have points of multiplicity $a-2$ along a smooth conic, and these are cones with vertex the plane of the conic. Hence

$$5\frac{(a+1)a}{2} - (h+1) = \frac{(a+1)(a-2)}{2},$$

so that

$$h = \dim(|aH - (a-1)\mathcal{E}|) = a(2a+3) = \frac{r(r+3)}{2}$$

that is the embedding dimension of $X_{a,a}$.

It is also interesting to notice that the degree of the exceptional divisor $\mathcal{E}$ in $X_{a,a}$ is

$$(aH - (a-1)\mathcal{E})^3 \cdot \mathcal{E} = 6(a-1)^2.$$

Moreover since $\dim(|aH - a\mathcal{E}|) = \frac{(a+3)a}{2}$, we see that the span of $\mathcal{E}$ in $X_{a,a}$ has dimension

$$a(2a+3) - \frac{(a+3)a}{2} - 1 = 3\frac{(a+1)a}{2} - 1.$$

In general $\mathcal{E}$ is swept out by a 1–dimensional family of $(a-1)$–Veronese surfaces. For $a = 2$, $\mathcal{E}$ is in fact a rational normal scroll threefold of degree 6 in a $\mathbb{P}^8$.

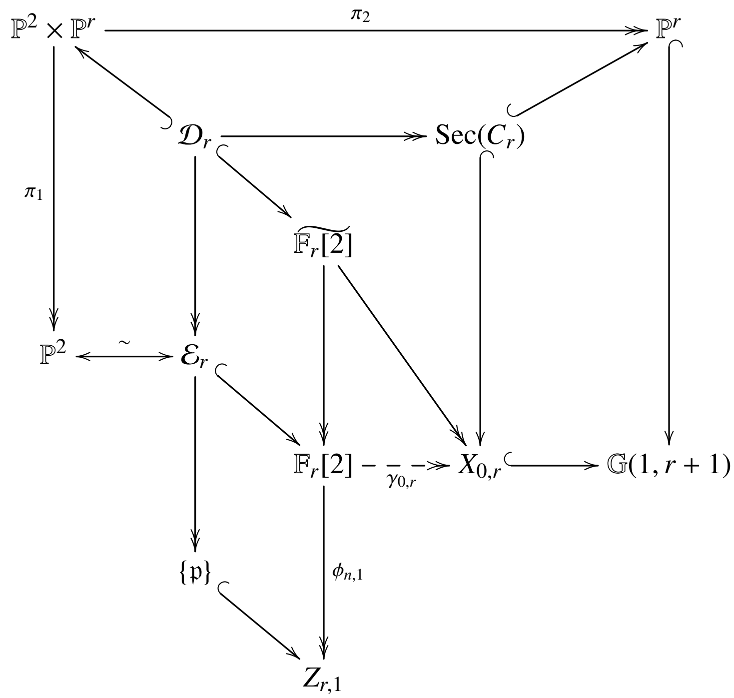
If we consider $S_{1,3} \cong \mathbb{F}_2$ the above picture changes. To put things in the general perspective we consider the following situation.

Extending the definition of $S_{a,b}$ we can consider $S_{0,r} \subset \mathbb{P}^{r+1}$ as the cone over the rational curve, image of $\mathbb{F}_r$ via the morphism determined by the linear system $|E + rF|$. Extending the definition of $\gamma_{a,b}$ we obtain a rational map

$$\gamma_{0,r} : \mathbb{F}_r[2] \dashrightarrow \mathbb{G}(1, r+1) \subset \mathbb{P}^{\frac{r(r+3)}{2}}$$

whose image we denote by $X_{0,r}$. The map is not defined exactly along the surface $\mathcal{E}_r \cong \mathbb{P}^2$ of the pair of points on E.

Resolving the indeterminacy we obtain the following commutative diagram, that we are going to explain.



Recall that the map $\phi_{n,1} \colon \mathbb{F}_r[2] \to Z_{r,1}$ is the contraction of $\mathcal{E}_r$ to a singular point $\mathfrak{p}$. A neighbourhood of $\mathfrak{p}$ in $Z_{r,1}$ is isomorphic to a neighbourhood U of the only singular point of $X_{1,r+1}$, and U was described in the proof of Theorem 3.7 as defined, in a chart of a Grassmannian, by the minors of order 2 of the $3 \times r$ matrix A analogous to the matrix in (3.5) (that was a $3 \times (r-2)$ matrix because we were describing $X_{1,r-1}$).

Each column of A defines a rational map $U \dashrightarrow \mathbb{P}^2$, map that does not depend on the choice of the column, undefined exactly at $\mathfrak{p}$. The reader can easily check that this map, undefined at $\mathfrak{p}$, lifts to a morphism on $\mathbb{F}_r[2]$, the morphism $\mathbb{F}_r[2] \to \mathbb{P}^1[2] \cong \mathbb{P}^2$ induced by the ruling $\mathbb{F}_r \to \mathbb{P}^1$. We blow up $Z_{r,1}$ at $\mathfrak{p}$. The exceptional divisor over $\mathfrak{p}$ is (Theorem 3.7) the smooth threefold linear section $\mathcal{D}_r$ of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^r$ in $\mathbb{P}^{3r+4}$ defined exactly by minors of order 2 of the matrix A. The columns of A define on $\mathcal{D}_r$ the projection π_1 on the first factor $\mathbb{P}^2$ which is then a lift of the analogous map just considered on $\mathbb{F}_r[2]$: this shows that the blow up of $Z_{r,1}$ at $\mathfrak{p}$ factors through $\mathbb{F}_r[2]$ and in fact coincides with the blow up $\widetilde{\mathbb{F}_r[2]}$ of $\mathbb{F}_r[2]$ at $\mathcal{E}_r$. This completes the description of the first three columns of the diagram.

Projecting $\mathcal{D}_r$ to the second factor, $\mathbb{P}^r$, we obtain a map that is a $\mathbb{P}^{3-r}$ bundle if $r = 1, 2$. If $r \geqslant 3$, $\mathcal{D}_r$ maps birationally to the secant variety of the rational normal curve. In fact $\gamma_{0,r}$, undefined at $\mathcal{E}_r$, lifts to a morphism on $\widetilde{\mathbb{F}_r[2]}$ that maps each point of $\mathcal{D}_r$ to a line through the singular point of $S_{0,r}$. The lines through that point form a $\mathbb{P}^r$ in $\mathbb{G}(1, r+1)$ and a simple explicit computation in coordinates shows that $\mathcal{D}_r$ maps in it as the second projection π_2.

It is worth noticing that the map $\gamma_{0,1}$ is a flip, whereas the map $\gamma_{0,2}$ is a *Mukai flop*, see [16].

So, when the smooth quadric $S_{1,1}$ flatly degenerates to the singular quadric $S_{0,2}$ (this degeneration can be realized in a linear pencil of quadrics in $\mathbb{P}^3$), we see that the pair $(\mathbb{G}(1,3),\Gamma')$ (with the conic Γ' not contained in a plane of $\mathbb{G}(1,3)$) to be blown up along Γ' to get $X_{2,2}$, degenerates to pair $(\mathbb{G}(1,3),\Gamma')$, with Γ' contained in a plane of $\mathbb{G}(1,3)$, to be blown up along Γ' to get $X_{1,3}$. The singularity of $X_{1,3}$ arises as the contraction to a point of the strict transform of the plane containing Γ'.

Remark 4.2. We notice that any FlF Y arises as the hyperplane section of $X_{1,3}$. Indeed, as we know, Y can be obtained in the following way. There is V, the general complete intersection of two quadrics Q and Q' in $\mathbb{P}^5$, and there is a smooth conic $\Gamma \subset V$, such that Y is the image in $\mathbb{P}^{13}$ via the 13–dimensional linear system $|\mathcal{I}_{\Gamma,V}(2)|$ of quadric sections of V containing Γ. Let Π be the plane containing Γ. In the pencil generated by Q and Q', there is a unique quadric Q_0 containing Π. The image of Q_0 in $\mathbb{P}^{14}$ via the 14–dimensional linear system $|\mathcal{I}_{\Gamma,Q_0}(2)|$ of quadric sections of Q_0 containing Γ is an $X_{1,3}$ having Y as a hyperplane section.

Remark 4.3. One can consider degenerations of $X_{2,2}$ and of $X_{1,3}$ in the following way. Consider a smooth quadric Q in $\mathbb{P}^5$ and a singular reduced conic Γ contained in Q. Then take the image X of Q in $\mathbb{P}^{14}$ via the 14–dimensional linear system $|\mathcal{I}_{\Gamma,Q}(2)|$ of quadric sections of Q containing Γ. If Γ is not contained in a plane of Q, X is a degeneration of $X_{2,2}$ that is singular along a line. If Γ is contained in a plane of Q, then X is also a degeneration of $X_{1,3}$.

Similarly, one can consider a singular quadric Q in $\mathbb{P}^5$ and a smooth conic Γ contained in the smooth locus of Q. Then take the image Z of Q in $\mathbb{P}^{14}$ via the 14–dimensional linear system $|\mathcal{I}_{\Gamma,Q}(2)|$ of quadric sections of Q containing Γ. Then Z is a singular degeneration of $X_{2,2}$ (and also of $X_{1,3}$, if the plane of Γ is contained in Q).

Arguing as in Remark 4.2, we see that a general FlF Z arises as the hyperplane section of such a X.

One can also consider degenerations of FlF to singular threefolds in various ways. We keep the notation of Remark 4.2. One possibility is to keep the complete intersection V of two quadrics Q and Q' in $\mathbb{P}^5$ smooth, but to take the conic Γ, to be blown up to get the FlF Y, singular but reduced. In this case Y has a simple double point.

Another possibility, suggested to us by Ivan Cheltsov, is to take V singular and the conic Γ still smooth. For example, we can take V as the image of $\mathbb{P}^3$ via the linear system of quadrics passing through four points $p_1,\dots,p_4 \in \mathbb{P}^3$ in general position. If we blow–up $\mathbb{P}^3$ at $p_1,\dots,p_4$, with exceptional divisors $E_1,\dots,E_4$, the above map becomes a morphism on this blow–up $\widetilde{\mathbb{P}^3}$, that maps $E_1,\dots,E_4$ to four planes and contracts the strict transforms of the six lines pairwise joining $p_1,\dots,p_4$ to six simple double points. There are many conics on V, for instance the images of general lines of $\mathbb{P}^3$. If we blow–up one of them we get a weak FlF double at six points.

5. A GIT analysis

As we noticed in Remark 4.2, any FlF is a hyperplane section of $X_{1,3} \subset \mathbb{P}^{14}$. Let G be the automorphism group G of $S_{1,3} = \mathbb{F}_2$. The group G has dimension 7 and it is isomorphic to the automorphism group of a quadric cone in $\mathbb{P}^3$. This is an extension of $\mathrm{PGL}(2,\mathbb{C})$ with the 4–dimensional normal subgroup G_0 of the projective transformation of $\mathbb{P}^3$ that fix a point x and also map every line of $\mathbb{P}^3$ through x to itself. The group G acts as a group of projective trasformations of $X_{1,3}$. Recall that $X_{1,3}$ has a unique singular point p and contains a rational normal quartic curve Γ that is the image of the lines of the ruling of $S_{1,3}$. Of course G fixes p and maps Γ to itself inducing on it the action of $\mathrm{PGL}(2,\mathbb{C})$. Moreover G acts in a natural way on the dual space $\mathcal{H} := (\mathbb{P}^{14})^\vee$ (and on the vector space $H^0(X_{1,3}, \mathcal{O}_{X_{1,3}}(1)))$ by acting on the hyperplane sections of $X_{1,3}$ that, if smooth, are FlFs. We want to understand the G–semistable elements of $\mathcal{H}$. A partial answer to this question is the following:

Proposition 5.1. *Let H be a hyperplane section of $X_{1,3}$ not containing the curve Γ and cutting out on Γ a divisor not containing a point with multiplicity 3. Then H is G–semistable.*

Proof. We argue by contradiction. Suppose that H is not semistable. Let $s \in H^0(X_{1,3}, \mathcal{O}_{X_{1,3}}(1))$ be a non–zero section, determined up to a constant, that vanishes on H. Then there is a sequence $\{g_n\}_{n\in\mathbb{N}}$ of elements of G such that $\lim_n g_n.s = 0$. But then, if σ is the restriction of s to Γ we also have $\lim_n g_n.\sigma = 0$ and this is a contradiction because the divisor cut out by H on Γ is $\mathrm{PGL}(2,\mathbb{C})$–semistable (see [14, Prop. 4.1]). □

The conclusion is that $\mathcal{H}^{ss}$ is non–empty, and therefore there exists the categorical quotient $\mathcal{M} = \mathcal{H}^{ss} /\!/ G$, that has dimension 7. Note that the Kuranishi family of a FlF has also dimension 7 (see [8, 4]). Hence locally we can consider $\mathcal{M}$ as a finite cover of the Kuranishi family. It follows that G is the component of the identity of the group of projective automorphisms of $X_{1,3}$ (we believe that in fact G is equal to the group of projective transformations of $X_{1,3}$ but we have not been able to prove it so far). Moreover we can consider somehow $\mathcal{M}$ as a moduli space of FlFs. However some crucial questions remains open. For example:
(i) are all smooth hyperplane sections of $X_{1,3}$ G (semi)stable?
(ii) are two smooth hyperplane sections of $X_{1,3}$ isomorphic if and only if they are G–isomorphic?
(iii) are FlFs K–stable?

An affirmative answer to question (iii) would provide a true moduli space for FlFs. We will not deal with these questions here.

References

[1] E. Arbarello, M. Cornalba, Ph. Griffiths, J. Harris, *Geometry of albebraic curves. Volume I*, Grundlehren der math. Wissenschaften, **267**, (1985), Springer–Verlag, Berlin Heidelberg.

[2] M. Andreatta, G. Occhetta, *Special rays in the Mori cone of a projective variety*, Nagoya Math. J., **168**, (2002), 127–137.

[3] M. Andreatta, R. Pignatelli, *Fano's last Fano*, Rend. Accad. Lincei Mat. Appl., **34** (2), (2023), 359–381.

[4] P. Belmans, E. Fatighenti, F. Tanturri, *Polyvector fields for Fano 3-folds*, Math. Z. **304**, (1), Paper No. 12, 30 pp.

[5] A. Bertram, I. Coskun, *The birational geometry of the Hilbert scheme of points on surfaces*, in *Birational geometry, rational curves, and arithmetic*, Springer Verlag, New York, 2013, 15–55.

[6] A. Cattaneo, *The degree of the tangent and secant variety to a projective surface*, Adv. Geom. **20**, (2), (2020), 233–248.

[7] G. Fano, *Su una particolare varietà algebrica a tre dimensioni aventi curve sezioni canoniche*, Rend. Acc. Naz. Lincei, **8** (6), (1949), 151–156.

[8] *Fanography* `https://www.fanography.info`

[9] J. Fogarty, *Algebraic Families on an Algebraic Surface, II, the Picard Scheme of the Punctual Hilbert Scheme*, Amer. J. Math., **95** (3), (1973), 660–687.

[10] A. Franchetta, *Sulla curva doppia della proiezione della superficie generale dell'S_4, da un punto generico su un S_3*, Rend. Acc. d'Italia, ser. VII, **2**, (1940), 282–288.

[11] A. Franchetta, *Sulla curva doppia della proiezione della superficie generale dell'S_4, da un punto generico su un S_3*, Lincei - Rend. Sc. fis. mat. e nat., **2**, (1947), 276–279.

[12] E. Mezzetti, D. Portelli, *A tour through some classical theorems on algebraic surfaces*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **5**, (2), (1997), 51–78.

[13] S. Mori, S. Mukai, *Classification of Fano 3–folds with $B_2 \geqslant 2$*, Manuscripta Math., **36**, (2), (1981/82), 147–162, Erratum, Manuscripta Math., **110**, (3), (2003), 407.

[14] D. Mumford, *Geometric Invariant Theory*, Springer Verlag, Ergebnisse der Math. und ihrer Grenzgeb., **34**, 1965.

[15] G. P. Pirola, E. Schlesinger, *Monodromy of projective curves*, J. Algebraic Geom., **14**, (4), (2005), 623–642.

[16] J. Wierzba, J. A. Wisniewski, *Small contractions of symplectic* 4*–folds*, Duke Math. J., **120**, (1), (2003), 65–95

[17] O. Zariski, *A theorem on the Poincaré group of an algebraic hypersurface*, Ann. of Math., **38**, (1937), 131–142.

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